

# THE FAILURE OF RATIONAL DILATION ON THE TETRABLOCK

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**ABSTRACT.** We show by a counter example that rational dilation fails on the tetrablock, a polynomially convex and non-convex domain in  $\mathbb{C}^3$  defined as

$$\mathbb{E} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : 1 - zx_1 - wx_2 + zwx_3 \neq 0 \text{ whenever } |z| \leq 1, |w| \leq 1\}.$$

A commuting triple of operators  $(T_1, T_2, T_3)$  for which the closed tetrablock  $\overline{\mathbb{E}}$  is a spectral set, is called an  $\mathbb{E}$ -contraction. For an  $\mathbb{E}$ -contraction  $(T_1, T_2, T_3)$ , the two operator equations

$$T_1 - T_2^* T_3 = D_{T_3} X_1 D_{T_3} \text{ and } T_2 - T_1^* T_3 = D_{T_3} X_2 D_{T_3}, \quad D_{T_3} = (I - T_3^* T_3)^{\frac{1}{2}},$$

have unique solutions  $A_1, A_2$  on  $\mathcal{D}_{T_3} = \overline{\text{Ran}} D_{T_3}$  and they are called the fundamental operators of  $(T_1, T_2, T_3)$ . For a particular class of  $\mathbb{E}$ -contractions, we prove it necessary for the existence of rational dilation that the corresponding fundamental operators  $A_1, A_2$  satisfy

$$A_1 A_2 = A_2 A_1 \text{ and } A_1^* A_1 - A_1 A_1^* = A_2^* A_2 - A_2 A_2^*. \quad (0.1)$$

Then we construct an  $\mathbb{E}$ -contraction from that particular class which fails to satisfy (0.1). We produce a concrete functional model for pure  $\mathbb{E}$ -isometries, a class of  $\mathbb{E}$ -contractions analogous to the pure isometries in one variable. The fundamental operators play the main role in this model.

## 1. INTRODUCTION

Let  $X$  be a compact subset of  $\mathbb{C}^n$  and let  $\mathcal{R}(X)$  denote the algebra of all rational functions on  $X$ , that is, all quotients  $p/q$  of polynomials  $p, q$  for which  $q$  has no zeros in  $X$ . The norm of an element  $f$  in  $\mathcal{R}(X)$  is defined as

$$\|f\|_{\infty, X} = \sup\{|f(\xi)| : \xi \in X\}.$$

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Also for each  $k \geq 1$ , let  $\mathcal{R}_k(X)$  denote the algebra of all  $k \times k$  matrices over  $\mathcal{R}(X)$ . Obviously each element in  $\mathcal{R}_k(X)$  is a  $k \times k$  matrix of rational functions  $F = (f_{i,j})$  and we can define a norm on  $\mathcal{R}_k(X)$  in the canonical way

$$\|F\| = \sup\{\|F(\xi)\| : \xi \in X\},$$

thereby making  $\mathcal{R}_k(X)$  into a non-commutative normed algebra. Let  $\underline{T} = (T_1, \dots, T_n)$  be an  $n$ -tuple of commuting operators on a Hilbert space  $\mathcal{H}$ . The set  $X$  is said to be a *spectral set* for  $\underline{T}$  if the Taylor joint spectrum  $\sigma(\underline{T})$  of  $\underline{T}$  is a subset of  $X$  and

$$\|f(\underline{T})\| \leq \|f\|_{\infty, X}, \text{ for every } f \in \mathcal{R}(X). \quad (1.1)$$

Here  $f(\underline{T})$  can be interpreted as  $p(\underline{T})q(\underline{T})^{-1}$  when  $f = p/q$ . Moreover,  $X$  is said to be a *complete spectral set* if  $\|F(\underline{T})\| \leq \|F\|$  for every  $F$  in  $\mathcal{R}_k(X)$ ,  $k = 1, 2, \dots$ .

Let  $\mathcal{A}(X)$  be the algebra of continuous complex-valued functions on  $X$  which separates the points of  $X$ . A *boundary* for  $\mathcal{A}(X)$  is a closed subset  $F$  of  $X$  such that every function in  $\mathcal{A}(X)$  attains its maximum modulus on  $F$ . It follows from the theory of uniform algebras that if  $bX$  is the intersection of all the boundaries of  $X$  then  $bX$  is a boundary for  $\mathcal{A}(X)$  (see Theorem 9.1 of [6]). This smallest boundary  $bX$  is called the *Šilov boundary* relative to the algebra  $\mathcal{A}(X)$ .

A commuting  $n$ -tuple of operators  $\underline{T}$  that has  $X$  as a spectral set, is said to have a *rational dilation* or *normal  $bX$ -dilation* if there exists a Hilbert space  $\mathcal{K}$ , an isometry  $V : \mathcal{H} \rightarrow \mathcal{K}$  and an  $n$ -tuple of commuting normal operators  $\underline{N} = (N_1, \dots, N_n)$  on  $\mathcal{K}$  with  $\sigma(\underline{N}) \subseteq bX$  such that

$$f(\underline{T}) = V^* f(\underline{N}) V, \text{ for every } f \in \mathcal{R}(X). \quad (1.2)$$

One of the important discoveries in operator theory is Sz.-Nagy's unitary dilation for a contraction, [20], which opened a new horizon by announcing the success of rational dilation on the closed unit disc of  $\mathbb{C}$ . Since then one of the main aims of operator theory has been to determine the success or failure of rational dilation on the closure of a bounded domain in  $\mathbb{C}^n$ . It is evident from the definitions that if  $X$  is a complete spectral set for  $\underline{T}$  then  $X$  is a spectral set for  $\underline{T}$ . A celebrated theorem of Arveson states that  $\underline{T}$  has a normal  $bX$ -dilation if and only if  $X$  is a complete spectral set for  $\underline{T}$  (Theorem 1.2.2 and its corollary, [8]). Therefore, the success or failure of rational dilation is equivalent to asking whether the fact that  $X$  is a spectral set for  $\underline{T}$  automatically turns  $X$  into a complete spectral set for  $\underline{T}$ . History

witnessed an affirmative answer to this question given by Agler when  $X$  is an annulus [3] and by Ando when  $X = \overline{\mathbb{D}^2}$  [7]. Agler, Harland and Raphael have produced, by machine computation, an example of a triply connected domain in  $\mathbb{C}$  where the answer is negative [4]. Dritschel and McCullough also gave a negative answer to that question when  $X$  is an arbitrary triply connected domain [12]. Parrott showed by a counter example [18] that rational dilation fails on the closed tridisc  $\overline{\mathbb{D}^3}$ . Also recently we have success of rational dilation on the closed symmetrized bidisc  $\Gamma$  [5, 10, 16], where  $\Gamma$  is defined as

$$\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1| \leq 1, |z_2| \leq 1\}. \quad (1.3)$$

In this article, we show that rational dilation fails when  $X$  is the closure of the tetrablock  $\mathbb{E}$ , a polynomially convex, non-convex and inhomogeneous domain in  $\mathbb{C}^3$ , defined as

$$\mathbb{E} = \{(x_1, x_2, x_3) \in \mathbb{C}^3 : 1 - zx_1 - wx_2 + zw x_3 \neq 0 \text{ whenever } |z| \leq 1, |w| \leq 1\}.$$

This domain has attracted the attention of a number of mathematicians [1, 2, 22, 13, 14, 23, 9, 11, 17] because of its relevance to  $\mu$ -synthesis and  $H^\infty$  control theory. The following result from [1] (Theorem 2.4, part-(9)) characterizes points in  $\mathbb{E}$  and  $\overline{\mathbb{E}}$  and provides a geometric description of the tetrablock.

**Theorem 1.1.** *A point  $(x_1, x_2, x_3) \in \mathbb{C}^3$  is in  $\overline{\mathbb{E}}$  if and only if  $|x_3| \leq 1$  and there exist  $\beta_1, \beta_2 \in \mathbb{C}$  such that  $|\beta_1| + |\beta_2| \leq 1$  and  $x_1 = \beta_1 + \bar{\beta}_2 x_3$ ,  $x_2 = \beta_2 + \bar{\beta}_1 x_3$ .*

It is evident from the above result that the tetrablock lives inside the tridisc  $\mathbb{D}^3$ . The distinguished boundary (which is same as the Šilov boundary) of the tetrablock was determined in [1] (see Theorem 7.1 of [1]) to be the set

$$\begin{aligned} b\overline{\mathbb{E}} &= \{(x_1, x_2, x_3) \in \mathbb{C}^3 : x_1 = \bar{x}_2 x_3, |x_2| \leq 1, |x_3| = 1\} \\ &= \{(x_1, x_2, x_3) \in \overline{\mathbb{E}} : |x_3| = 1\}. \end{aligned}$$

In [9], Bhattacharyya introduced the study of commuting operator triples that have  $\overline{\mathbb{E}}$  as a spectral set. There such a triple was called a *tetrablock contraction*. As a notation is always convenient, we shall call such a triple an  $\mathbb{E}$ -contraction. So we are led to the following definition:

**Definition 1.2.** A triple of commuting operators  $(T_1, T_2, T_3)$  on a Hilbert space  $\mathcal{H}$  for which  $\overline{\mathbb{E}}$  is a spectral set is called an  $\mathbb{E}$ -contraction.

Since the tetrablock lives inside the tridisc, an  $\mathbb{E}$ -contraction consists of commuting contractions. Evidently  $(T_1^*, T_2^*, T_3^*)$  is an  $\mathbb{E}$ -contraction

when  $(T_1, T_2, T_3)$  is an  $\mathbb{E}$ -contraction. We briefly recall from the literature the special classes of  $\mathbb{E}$ -contractions which are analogous to uniteries, isometries and co-isometries in one variable operator theory.

**Definition 1.3.** Let  $T_1, T_2, T_3$  be commuting operators on a Hilbert space  $\mathcal{H}$ . We say that  $(T_1, T_2, T_3)$  is

- (i) an  $\mathbb{E}$ -unitary if  $T_1, T_2, T_3$  are normal operators and the joint spectrum  $\sigma_T(T_1, T_2, T_3)$  is contained in  $b\overline{\mathbb{E}}$  ;
- (ii) an  $\mathbb{E}$ -isometry if there exists a Hilbert space  $\mathcal{K}$  containing  $\mathcal{H}$  and an  $\mathbb{E}$ -unitary  $(\tilde{T}_1, \tilde{T}_2, \tilde{T}_3)$  on  $\mathcal{K}$  such that  $\mathcal{H}$  is a common invariant subspace of  $T_1, T_2, T_3$  and that  $T_i = \tilde{T}_i|_{\mathcal{H}}$  for  $i = 1, 2, 3$ ;
- (iii) an  $\mathbb{E}$ -co-isometry if  $(T_1^*, T_2^*, T_3^*)$  is an  $\mathbb{E}$ -isometry.

Moreover, an  $\mathbb{E}$ -isometry  $(T_1, T_2, T_3)$  is said to be *pure* if  $T_3$  is a pure isometry, i.e, if  $T_3^{*n} \rightarrow 0$  strongly as  $n \rightarrow \infty$ . We accumulate some results from the literature in section 2 and they will be used in sequel.

It is clear that a rational dilation of an  $\mathbb{E}$ -contraction  $(T_1, T_2, T_3)$  is nothing but an  $\mathbb{E}$ -unitary dilation of  $(T_1, T_2, T_3)$ , that is, an  $\mathbb{E}$ -unitary  $N = (N_1, N_2, N_3)$  that dilates  $T$  by satisfying (1.2). Similarly an  $\mathbb{E}$ -isometric dilation of  $T = (T_1, T_2, T_3)$  is an  $\mathbb{E}$ -isometry  $V = (V_1, V_2, V_3)$  that satisfies (1.2). In Theorem 3.5 in [9], it was shown that for every  $\mathbb{E}$ -contraction  $(T_1, T_2, T_3)$  there were two unique operators  $A_1, A_2$  in  $\mathcal{L}(\mathcal{D}_{T_3})$  such that

$$T_1 - T_2^* T_3 = D_{T_3} A_1 D_{T_3}, \quad T_2 - T_1^* T_3 = D_{T_3} A_2 D_{T_3}.$$

Here  $D_{T_3} = (I - T_3^* T_3)^{\frac{1}{2}}$  and  $\mathcal{D}_{T_3} = \overline{Ran} D_{T_3}$  and  $\mathcal{L}(\mathcal{H})$ , for a Hilbert space  $\mathcal{H}$ , always denotes the algebra of bounded operators on  $\mathcal{H}$ . An explicit  $\mathbb{E}$ -isometric dilation was constructed for a particular class of  $\mathbb{E}$ -contractions in [9] (see Theorem 6.1 in [9]) and  $A_1, A_2$  played the fundamental role in that explicit construction of dilation. For their pivotal role in the dilation,  $A_1$  and  $A_2$  were called the *fundamental operators* of  $(T_1, T_2, T_3)$ .

In section 4, we produce a set of necessary conditions for the existence of rational dilation for a class of  $\mathbb{E}$ -contractions. Indeed, in Proposition 4.5, we show that if  $(T_1, T_2, T_3)$  is an  $\mathbb{E}$ -contraction on  $\mathcal{H}_1 \oplus \mathcal{H}_1$  for some Hilbert space  $\mathcal{H}_1$ , satisfying

- (i)  $Ker(D_{T_3}) = \mathcal{H}_1 \oplus \{0\}$  and  $\mathcal{D}_{T_3} = \{0\} \oplus \mathcal{H}_1$
- (ii)  $T_3(\mathcal{D}_{T_3}) = \{0\}$  and  $T_3 Ker(D_{T_3}) \subseteq \mathcal{D}_{T_3}$

and if  $A_1, A_2$  are the fundamental operators of  $(T_1, T_2, T_3)$ , then for the existence of an  $\mathbb{E}$ -isometric dilation of  $(T_1^*, T_2^*, T_3^*)$  it is necessary that

$$[A_1, A_2] = 0 \text{ and } [A_1^*, A_1] = [A_2^*, A_2]. \quad (1.4)$$

Here  $[S_1, S_2] = S_1 S_2 - S_2 S_1$ , for any two operators  $S_1, S_2$ . In section 5, we construct an example of an  $\mathbb{E}$ -contraction that satisfies the hypotheses of Proposition 4.5 but fails to satisfy (1.4). This concludes the failure of rational dilation on the tetrablock.

The proof of Proposition 4.5 depends heavily upon a functional model for pure  $\mathbb{E}$ -isometries which we provide in Theorem 3.3. There is an Wold type decomposition for an  $\mathbb{E}$ -isometry (see Theorem 2.3) that splits an  $\mathbb{E}$ -isometry into two parts of which one is an  $\mathbb{E}$ -unitary and the other is a pure  $\mathbb{E}$ -isometry. Theorem 2.2 describes the structure of an  $\mathbb{E}$ -unitary. Therefore, a concrete model for pure  $\mathbb{E}$ -isometries gives a complete description of an  $\mathbb{E}$ -isometry. In Theorem 3.3, we show that a pure  $\mathbb{E}$ -isometry  $(\hat{T}_1, \hat{T}_2, \hat{T}_3)$  can be modelled as a commuting triple of Toeplitz operators  $(T_{A_1^* + A_2 z}, T_{A_2^* + A_1 z}, T_z)$  on the vectorial Hardy space  $H^2(\mathcal{D}_{\hat{T}_3^*})$ , where  $A_1$  and  $A_2$  are the fundamental operators of the  $\mathbb{E}$ -coisometry  $(\hat{T}_1^*, \hat{T}_2^*, \hat{T}_3^*)$ . The converse is also true, that is, every such triple of commuting contractions  $(T_{A+Bz}, T_{B^*+A^*z}, T_z)$  on a vectorial Hardy space is a pure  $\mathbb{E}$ -isometry.

## 2. PRELIMINARY RESULTS

We begin with a lemma that simplifies the definition of  $\mathbb{E}$ -contraction.

**Lemma 2.1.** *A commuting triple of bounded operators  $(T_1, T_2, T_3)$  is an  $\mathbb{E}$ -contraction if and only if  $\|f(T_1, T_2, T_3)\| \leq \|f\|_{\infty, \mathbb{E}}$  for any holomorphic polynomial  $f$  in three variables.*

This actually follows from the fact that  $\overline{\mathbb{E}}$  is polynomially convex. For a proof to this lemma see Lemma 3.3 of [9]. The following theorem gives a set of characterization for  $\mathbb{E}$ -unitaries (Theorem 5.4 of [9]).

**Theorem 2.2.** *Let  $\underline{N} = (N_1, N_2, N_3)$  be a commuting triple of bounded operators. Then the following are equivalent.*

- (1)  $\underline{N}$  is an  $\mathbb{E}$ -unitary,
- (2)  $N_3$  is a unitary and  $\underline{N}$  is an  $\mathbb{E}$ -contraction,
- (3)  $N_3$  is a unitary,  $N_2$  is a contraction and  $N_1 = N_2^* N_3$ .

Here is a structure theorem for the  $\mathbb{E}$ -isometries.

**Theorem 2.3.** *Let  $\underline{V} = (V_1, V_2, V_3)$  be a commuting triple of bounded operators. Then the following are equivalent.*

- (1)  $\underline{V}$  is an  $\mathbb{E}$ -isometry.
- (2)  $V_3$  is an isometry and  $\underline{V}$  is an  $\mathbb{E}$ -contraction.
- (3)  $V_3$  is an isometry,  $V_2$  is a contraction and  $V_1 = V_2^*V_3$ .
- (4) (Wold decomposition)  $\mathcal{H}$  has a decomposition  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$  into reducing subspaces of  $V_1, V_2, V_3$  such that  $(V_1|_{\mathcal{H}_1}, V_2|_{\mathcal{H}_1}, V_3|_{\mathcal{H}_1})$  is an  $\mathbb{E}$ -unitary and  $(V_1|_{\mathcal{H}_2}, V_2|_{\mathcal{H}_2}, V_3|_{\mathcal{H}_2})$  is a pure  $\mathbb{E}$ -isometry.

See Theorem 5.6 and Theorem 5.7 of [9] for a proof.

### 3. A FUNCTIONAL MODEL FOR PURE $E$ -ISOMETRIES

Let us recall that the *numerical radius* of an operator  $T$  on a Hilbert space  $\mathcal{H}$  is defined by

$$\omega(T) = \sup\{|\langle Tx, x \rangle| : \|x\|_{\mathcal{H}} = 1\}.$$

It is well known that

$$r(T) \leq \omega(T) \leq \|T\| \text{ and } \frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|, \quad (3.1)$$

where  $r(T)$  is the spectral radius of  $T$ . We state a basic lemma on numerical radius whose proof is a routine exercise. We shall use this lemma in sequel.

**Lemma 3.1.** *The numerical radius of an operator  $T$  is not greater than one if and only if  $\operatorname{Re} \beta T \leq I$  for all complex numbers  $\beta$  of modulus 1.*

We recall from section 1, the existence-uniqueness theorem ([9], Theorem 3.5) for the fundamental operators of an  $\mathbb{E}$ -contraction.

**Theorem 3.2.** *Let  $(T_1, T_2, T_3)$  be an  $\mathbb{E}$ -contraction. Then there are two unique operators  $A_1, A_2$  in  $\mathcal{L}(\mathcal{D}_{T_3})$  such that*

$$T_1 - T_2^*T_3 = D_{T_3}A_1D_{T_3} \text{ and } T_2 - T_1^*T_3 = D_{T_3}A_2D_{T_3}. \quad (3.2)$$

Moreover,  $\omega(A_1 + zA_2) \leq 1$  for all  $z \in \overline{\mathbb{D}}$ .

As we mentioned in Section 1 that these two unique operators  $A_1, A_2$  are called the fundamental operators of  $(T_1, T_2, T_3)$ . The following theorem gives a concrete model for pure  $\mathbb{E}$ -isometries in terms of Toeplitz operators on a vectorial Hardy space.

**Theorem 3.3.** *Let  $(\hat{T}_1, \hat{T}_2, \hat{T}_3)$  be a pure  $\mathbb{E}$ -isometry acting on a Hilbert space  $\mathcal{H}$  and let  $A_1, A_2$  denote the corresponding fundamental operators. Then there exists a unitary  $U : \mathcal{H} \rightarrow H^2(\mathcal{D}_{\hat{T}_3}^*)$  such that*

$$\hat{T}_1 = U^*T_\varphi U, \quad \hat{T}_2 = U^*T_\psi U \text{ and } \hat{T}_3 = U^*T_z U,$$

where  $\varphi(z) = G_1^* + G_2z$ ,  $\psi(z) = G_2^* + G_1z$ ,  $z \in \mathbb{D}$  and  $G_1 = UA_1U^*$  and  $G_2 = UA_2U^*$ . Moreover,  $A_1, A_2$  satisfy

- (1)  $[A_1, A_2] = 0$ ;
- (2)  $[A_1^*, A_1] = [A_2^*, A_2]$ ; and
- (3)  $\|A_1^* + A_2 z\| \leq 1$  for all  $z \in \mathbb{D}$ .

Conversely, if  $A_1$  and  $A_2$  are two bounded operators on a Hilbert space  $E$  satisfying the above three conditions, then  $(T_{A_1^* + A_2 z}, T_{A_2^* + A_1 z}, T_z)$  on  $H^2(E)$  is a pure  $\mathbb{E}$ -isometry.

*Proof.* Suppose that  $(\hat{T}_1, \hat{T}_2, \hat{T}_3)$  is a pure  $\mathbb{E}$ -isometry. Then  $\hat{T}_3$  is a pure isometry and it can be identified with the Toeplitz operator  $T_z$  on  $H^2(\mathcal{D}_{\hat{T}_3}^*)$ . Therefore, there is a unitary  $U$  from  $\mathcal{H}$  onto  $H^2(\mathcal{D}_{\hat{T}_3}^*)$  such that  $\hat{T}_3 = U^* T_z U$ . Since for  $i = 1, 2$ ,  $\hat{T}_i$  is a commutant of  $\hat{T}_3$ , there are two multipliers  $\varphi, \psi$  in  $H^\infty(\mathcal{L}(\mathcal{D}_{\hat{T}_3}^*))$  such that  $\hat{T}_1 = U^* T_\varphi U$  and  $\hat{T}_2 = U^* T_\psi U$ .

*Claim.* If  $(V_1, V_2, V_3)$  on a Hilbert space  $\mathcal{H}_1$  is an  $\mathbb{E}$ -isometry then  $V_2 = V_1^* V_3$ .

*Proof of Claim.* Let  $(V_1, V_2, V_3)$  be the restriction of an  $\mathbb{E}$ -unitary  $(N_1, N_2, N_3)$  to the common invariant subspace  $\mathcal{H}_1$ . By part-(3) of Theorem 2.2 we have that  $N_3$  is a unitary and  $N_1 = N_2^* N_3$ . Therefore,  $N_1^* = N_3^* N_2$  and hence  $N_1^* = N_2 N_3^*$  by an application of Fuglede's theorem, [15], which states that if a normal operator  $N$  commutes with a bounded operator  $T$  then it commutes with  $T^*$  too. Also since  $N_3$  is a unitary we have that  $N_2 = N_1^* N_3$ . Now  $\mathcal{H}_1$  is an invariant subspace for  $N_2$  and thus  $\mathcal{H}_1$  is invariant under  $N_1^* N_3$ . So  $V_2 = N_2|_{\mathcal{H}_1} = N_1^* N_3|_{\mathcal{H}_1}$ . Again  $\mathcal{H}_1$  is invariant under  $N_3$ . Therefore,  $N_1^*(N_3(\mathcal{H}_1)) \subseteq \mathcal{H}_1$ . So we have that  $P_{\mathcal{H}_1} N_1^*|_{N_3(\mathcal{H}_1)} = N_1^*|_{N_3(\mathcal{H}_1)}$ . Again  $V_1^* = P_{\mathcal{H}_1} N_1^*|_{\mathcal{H}_1}$ . Therefore,  $N_1^* N_3|_{\mathcal{H}_1} = V_1^* V_3$ . So, we have that  $V_2 = V_1^* V_3$ .

We apply this claim and part-(3) of Theorem 2.3 to the  $\mathbb{E}$ -isometry  $(T_\varphi, T_\psi, T_z)$  to get  $T_\varphi = T_\psi^* T_z$  and  $T_\psi = T_\varphi^* T_z$  and by these two relations we have that

$$\varphi(z) = G_1^* + G_2 z \text{ and } \psi(z) = G_2^* + G_1 z \text{ for some } G_1, G_2 \in \mathcal{L}(\mathcal{D}_{\hat{T}_3}^*).$$

By the commutativity of  $\varphi(z)$  and  $\psi(z)$  we obtain

$$[G_1, G_2] = 0 \text{ and } [G_1^*, G_1] = [G_2^*, G_2].$$

We now compute the fundamental operators of the  $\mathbb{E}$ -co-isometry  $(T_\varphi^*, T_\psi^*, T_z^*)$  that is of  $(T_{G_1^* + G_2 z}^*, T_{G_2^* + G_1 z}^*, T_z^*)$ . Clearly  $I - T_z T_z^*$  is the projection onto the space  $\mathcal{D}_{T_z^*}$ . Now

$$T_{G_1^* + G_2 z}^* - T_{G_2^* + G_1 z}^* T_z^* = T_{G_1 + G_2 \bar{z}} - T_{G_2^* + G_1 z} T_{\bar{z}} = T_{G_1} = (I - T_z T_z^*) G_1 (I - T_z T_z^*).$$

Similarly,

$$T_{G_2^*+G_1z}^* - T_{G_1^*+G_2z}T_z^* = (I - T_zT_z^*)G_2(I - T_zT_z^*).$$

Therefore,  $G_1, G_2$  are the fundamental operators of  $(T_\varphi^*, T_\psi^*, T_z^*)$ . The fundamental operators of  $(\hat{T}_1^*, \hat{T}_2^*, \hat{T}_3^*)$  are  $A_1, A_2$ . Therefore

$$\hat{T}_1^* - \hat{T}_2\hat{T}_3^* = D_{\hat{T}_3^*}A_1D_{\hat{T}_3^*}$$

that is

$$U^*(T_\varphi^* - T_\psi T_z^*)U = U^*D_{T_z^*}(UA_1U^*)D_{T_z^*}^*U$$

or equivalently

$$T_\varphi^* - T_\psi T_z^* = D_{T_z^*}(UA_1U^*)D_{T_z^*}^*.$$

Similarly,

$$T_\psi^* - T_\varphi T_z^* = D_{T_z^*}(UA_2U^*)D_{T_z^*}^*.$$

Therefore, by the uniqueness of fundamental operators (see Theorem 3.2) we have that

$$G_1 = UA_1U^* \text{ and } G_2 = UA_2U^*.$$

From  $[G_1, G_2] = 0$  and  $[G_1^*, G_1] = [G_2^*, G_2]$  it trivially follows that  $[A_1, A_2] = 0$  and  $[A_1^*, A_1] = [A_2^*, A_2]$ . Also since  $(T_\varphi, T_\psi, T_z)$  is an  $\mathbb{E}$ -contraction, we have that  $\|T_\varphi\| \leq 1$  and hence  $\|\varphi(z)\| = \|G_1^* + G_2z\| \leq 1$  for all  $z \in \mathbb{D}$ . Therefore,  $\|A_1^* + A_2z\| = \|U^*(G_1^* + G_2)U\| \leq 1$  for all  $z \in \mathbb{D}$ .

For the converse, we first prove that the triple of multiplication operators  $(M_{A_1^*+A_2z}, M_{A_2^*+A_1z}, M_z)$  on  $L^2(E)$  is an  $\mathbb{E}$ -unitary when  $A_1, A_2$  satisfy the given conditions. It is evident that  $(M_{A_1^*+A_2z}, M_{A_2^*+A_1z}, M_z)$  is a commuting triple of normal operators when  $[A_1, A_2] = 0$  and  $[A_1^*, A_1] = [A_2^*, A_2]$ . Also  $M_{A_1^*+A_2z} = M_{A_2^*+A_1z}^*M_z$  and  $M_z$  on  $L^2(E)$  is unitary. Therefore, by part-(3) of Theorem 2.2,  $(M_{A_1^*+A_2z}, M_{A_2^*+A_1z}, M_z)$  becomes an  $\mathbb{E}$ -unitary if we prove that  $\|M_{A_2^*+A_1z}\| \leq 1$  for all  $z \in \mathbb{T}$ . We have that  $\omega(A_1 + zA_2) \leq 1$  for every  $z \in \mathbb{T}$ , which is same as saying that  $\omega(z_1A_1 + z_2A_2) \leq 1$  for all complex numbers  $z_1, z_2$  of unit modulus. Thus by Lemma 3.1,

$$(z_1A_1 + z_2A_2) + (z_1A_1 + z_2A_2)^* \leq 2I,$$

that is

$$(z_1A_1 + \bar{z}_2A_2^*) + (z_1A_1 + \bar{z}_2A_2^*)^* \leq 2I.$$

Therefore,  $\bar{z}_2(A_2^* + zA_1) + z_2(A_2^* + zA_1)^* \leq 2I$  for all  $z, z_2 \in \mathbb{T}$ . This is same as saying that

$$\operatorname{Re} z_2(A_2^* + zA_1) \leq I, \text{ for all } z, z_2 \in \mathbb{T}.$$



Therefore, by Lemma 3.1 again  $\omega(A_2^* + A_1z) \leq 1$  for any  $z$  in  $\mathbb{T}$ . Since  $M_{A_2^*+A_1z}$  is a normal operator we have that  $\|M_{A_2^*+A_1z}\| = \omega(M_{A_2^*+A_1z})$  and thus  $\|M_{A_2^*+A_1z}\|$  for all  $z \in \mathbb{T}$ . Therefore,  $(M_{A_1^*+A_2z}, M_{A_2^*+A_1z}, M_z)$  on  $L^2(E)$  is an  $\mathbb{E}$ -unitary and hence  $(T_{A_1^*+A_2z}, T_{A_2^*+A_1z}, T_z)$ , being the restriction of  $(M_{A_1^*+A_2z}, M_{A_2^*+A_1z}, M_z)$  to the common invariant subspace  $H^2(E)$ , is an  $\mathbb{E}$ -isometry. Also  $T_z$  on  $H^2(E)$  is a pure isometry. Thus we conclude that  $(T_{A_1^*+A_2z}, T_{A_2^*+A_1z}, T_z)$  is a pure  $\mathbb{E}$ -isometry.  $\blacksquare$

#### 4. A NECESSARY CONDITION FOR THE EXISTENCE OF DILATION

Let us recall from section 1 the definitions of the  $\mathbb{E}$ -isometric and  $\mathbb{E}$ -unitary dilations of an  $\mathbb{E}$ -contraction. In fact they can be defined in a simpler way by involving polynomials only. This is because the polynomials are dense in the rational functions.

**Definition 4.1.** Let  $(T_1, T_2, T_3)$  be a  $\mathbb{E}$ -contraction on  $\mathcal{H}$ . A commuting tuple  $(Q_1, Q_2, V)$  on  $\mathcal{K}$  is said to be an  $\mathbb{E}$ -isometric dilation of  $(T_1, T_2, T_3)$  if  $\mathcal{H} \subseteq \mathcal{K}$ ,  $(Q_1, Q_2, V)$  is an  $\mathbb{E}$ -isometry and

$$P_{\mathcal{H}}(Q_1^{m_1} Q_2^{m_2} V^n)|_{\mathcal{H}} = T_1^{m_1} T_2^{m_2} T_3^n, \quad \text{for all non-negative integers } m_1, m_2, n.$$

Here  $P_{\mathcal{H}} : \mathcal{K} \rightarrow \mathcal{H}$  is the orthogonal projection of  $\mathcal{K}$  onto  $\mathcal{H}$ . Moreover, the dilation is called *minimal* if

$$\mathcal{K} = \overline{\text{span}}\{Q_1^{m_1} Q_2^{m_2} V^n h : h \in \mathcal{H} \text{ and } m_1, m_2, n \in \mathbb{N} \cup \{0\}\}.$$

**Definition 4.2.** A commuting tuple  $(R_1, R_2, U)$  on  $\mathcal{K}$  is said to be an  $\mathbb{E}$ -unitary dilation of  $(T_1, T_2, T_3)$  if  $\mathcal{H} \subseteq \mathcal{K}$ ,  $(R_1, R_2, U)$  is an  $\mathbb{E}$ -unitary and

$$P_{\mathcal{H}}(R_1^{m_1} R_2^{m_2} U^n)|_{\mathcal{H}} = T_1^{m_1} T_2^{m_2} T_3^n, \quad \text{for all non-negative integers } m_1, m_2, n.$$

Moreover, the dilation is called *minimal* if

$$\mathcal{K} = \overline{\text{span}}\{R_1^{m_1} R_2^{m_2} U^n h : h \in \mathcal{H} \text{ and } m_1, m_2, n \in \mathbb{Z}\}.$$

Here  $R_i^{m_i} = R_i^{*-m_i}$  for  $i = 1, 2$  and  $U^n = U^{*-n}$  when  $m_i$  and  $n$  are negative integers.

**Proposition 4.3.** *If a  $\mathbb{E}$ -contraction  $(T_1, T_2, T_3)$  defined on  $\mathcal{H}$  has a  $\mathbb{E}$ -isometric dilation, then it has a minimal  $\mathbb{E}$ -isometric dilation.*

*Proof.* Let  $(Q_1, Q_2, V)$  on  $\mathcal{K} \supseteq \mathcal{H}$  be a  $\mathbb{E}$ -isometric dilation of  $(T_1, T_2, T_3)$ . Let  $\mathcal{K}_0$  be the space defined as

$$\mathcal{K}_0 = \overline{\text{span}}\{Q_1^{m_1} Q_2^{m_2} V^n h : h \in \mathcal{H} \text{ and } m_1, m_2, n \in \mathbb{N} \cup \{0\}\}.$$

Clearly  $\mathcal{K}_0$  is invariant under  $Q_1^{m_1}, Q_2^{m_2}$  and  $V^n$ , for any non-negative integer  $m_1, m_2$  and  $n$ . Therefore if we denote the restrictions of  $Q_1, Q_2$

and  $V$  to the common invariant subspace  $\mathcal{K}_0$  by  $Q_{11}, Q_{12}$  and  $V_1$  respectively, we get

$$Q_{11}^{m_1}k = Q_1^{m_1}k, Q_{12}^{m_2}k = Q_2^{m_2}k, \text{ and } V_1^n k = V^n k, \quad \text{for any } k \in \mathcal{K}_0.$$

Hence

$$\mathcal{K}_0 = \overline{\text{span}}\{Q_{11}^{m_1}Q_{12}^{m_2}V_1^n h : h \in \mathcal{H} \text{ and } m_1, m_2, n \in \mathbb{N} \cup \{0\}\}.$$

Therefore for any non-negative integers  $m_1, m_2$  and  $n$  we have that

$$P_{\mathcal{H}}(Q_{11}^{m_1}Q_{12}^{m_2}V_1^n)h = P_{\mathcal{H}}(Q_1^{m_1}Q_2^{m_2}V^n)h, \quad \text{for all } h \in \mathcal{H}.$$

Now  $(Q_{11}, Q_{12}, V_1)$  is an  $\mathbb{E}$ -contraction by being the restriction of an  $\mathbb{E}$ -contraction  $(Q_1, Q_2, V)$  to a common invariant subspace  $\mathcal{K}_0$ . Also  $V_1$ , being the restriction of an isometry to an invariant subspace, is also an isometry. Therefore by Theorem 2.3 - part(2),  $(Q_{11}, Q_{12}, V_1)$  is an  $\mathbb{E}$ -isometry. Hence  $(Q_{11}, Q_{12}, V_1)$  is a minimal  $\mathbb{E}$ -isometric dilation of  $(T_1, T_2, T_3)$ . ■

**Proposition 4.4.** *Let  $(Q_1, Q_2, V)$  on  $\mathcal{K}$  be an  $\mathbb{E}$ -isometric dilation of an  $\mathbb{E}$ -contraction  $(T_1, T_2, T_3)$  on  $\mathcal{H}$ . If  $(Q_1, Q_2, V)$  is minimal, then  $(Q_1^*, Q_2^*, V^*)$  is an  $\mathbb{E}$ -co-isometric extension of  $(T_1^*, T_2^*, T_3^*)$ .*

*Proof.* We first prove that  $T_1 P_{\mathcal{H}} = P_{\mathcal{H}} Q_1, T_2 P_{\mathcal{H}} = P_{\mathcal{H}} Q_2$  and  $T_3 P_{\mathcal{H}} = P_{\mathcal{H}} V$ . Clearly

$$\mathcal{K} = \overline{\text{span}}\{Q_1^{m_1}Q_2^{m_2}V^n h : h \in \mathcal{H} \text{ and } m_1, m_2, n \in \mathbb{N} \cup \{0\}\}.$$

Now for  $h \in \mathcal{H}$  we have that

$$\begin{aligned} T_1 P_{\mathcal{H}}(Q_1^{m_1}Q_2^{m_2}V^n h) &= T_1(T_1^{m_1}T_2^{m_2}T_3^n h) = T_1^{m_1+1}T_2^{m_2}T_3^n h = P_{\mathcal{H}}(Q_1^{m_1+1}Q_2^{m_2}V^n h) \\ &= P_{\mathcal{H}}Q_1(Q_1^{m_1}Q_2^{m_2}V^n h). \end{aligned}$$

Thus we have that  $T_1 P_{\mathcal{H}} = P_{\mathcal{H}} Q_1$  and similarly we can prove that  $T_2 P_{\mathcal{H}} = P_{\mathcal{H}} Q_2$  and  $T_3 P_{\mathcal{H}} = P_{\mathcal{H}} V$ . Also for  $h \in \mathcal{H}$  and  $k \in \mathcal{K}$  we have that

$$\langle T_1^* h, k \rangle = \langle P_{\mathcal{H}} T_1^* h, k \rangle = \langle T_1^* h, P_{\mathcal{H}} k \rangle = \langle h, T_1 P_{\mathcal{H}} k \rangle = \langle h, P_{\mathcal{H}} Q_1 k \rangle = \langle Q_1^* h, k \rangle.$$

Hence  $T_1^* = Q_1^*|_{\mathcal{H}}$  and similarly  $T_2^* = Q_2^*|_{\mathcal{H}}$  and  $T_3^* = V^*|_{\mathcal{H}}$ . Therefore,  $(Q_1^*, Q_2^*, V^*)$  is an  $\mathbb{E}$ -co-isometric extension of  $(T_1^*, T_2^*, T_3^*)$ . ■

**Proposition 4.5.** *Let  $\mathcal{H}_1$  be a Hilbert space and let  $(T_1, T_2, T_3)$  be an  $\mathbb{E}$ -contraction on  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$  with fundamental operators  $A_1, A_2$ . Let*

- (i)  $\text{Ker}(D_{T_3}) = \mathcal{H}_1 \oplus \{0\}$  and  $\mathcal{D}_{T_3} = \{0\} \oplus \mathcal{H}_1$ ;
- (ii)  $T_3(\mathcal{D}_{T_3}) = \{0\}$  and  $T_3 \text{Ker}(D_{T_3}) \subseteq \mathcal{D}_{T_3}$ .

*If  $(T_1^*, T_2^*, T_3^*)$  has an  $\mathbb{E}$ -isometric dilation then*

- (1)  $A_1 A_2 = A_2 A_1$ ,
- (2)  $A_1^* A_1 - A_1 A_1^* = A_2^* A_2 - A_2 A_2^*$ .

*Proof.* Let  $(Q_1, Q_2, V)$  on a Hilbert space  $\mathcal{K}$  be a minimal  $\mathbb{E}$ -isometric dilation of  $(T_1^*, T_2^*, T_3^*)$  (such a minimal  $\mathbb{E}$ -isometric dilation exists by Proposition 4.3) so that  $(Q_1^*, Q_2^*, V^*)$  is an  $\mathbb{E}$ -co-isometric extension of  $(T_1, T_2, T_3)$  by Proposition 4.4. Since  $(Q_1, Q_2, V)$  on  $\mathcal{K}$  is an  $E$ -isometry, by part-(4) of Theorem 2.3,  $\mathcal{K}$  has decomposition  $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$  into reducing subspaces  $\mathcal{K}_1, \mathcal{K}_2$  of  $Q_1, Q_2, V$  such that  $(Q_1|_{\mathcal{K}_1}, Q_2|_{\mathcal{K}_1}, V|_{\mathcal{K}_1}) = (Q_{11}, Q_{12}, U_1)$  is an  $\mathbb{E}$ -unitary and  $(Q_1|_{\mathcal{K}_2}, Q_2|_{\mathcal{K}_2}, V|_{\mathcal{K}_2}) = (Q_{21}, Q_{22}, V_1)$  is a pure  $\mathbb{E}$ -isometry. Since  $(Q_{21}, Q_{22}, V_1)$  on  $\mathcal{K}_2$  is a pure  $\mathbb{E}$ -isometry, by Theorem 3.3,  $\mathcal{K}_2$  can be identified with  $H^2(E)$ , where  $E = \mathcal{D}_{V_1^*}$  and  $Q_{21}, Q_{22}, V_1$  can be identified with  $T_\varphi, T_\psi, T_z$  respectively on  $H^2(E)$ , where  $\varphi(z) = A + Bz$  and  $\psi(z) = B^* + A^*z$ ,  $z \in \mathbb{D}$ . Here  $A^*, B$  are the fundamental operators of  $(Q_{21}^*, Q_{22}^*, V_1^*)$ . Again  $H^2(E)$  can be identified with  $l^2(E)$  and  $T_\varphi, T_\psi, T_z$  on  $H^2(E)$  can be identified with the multiplication operators  $M_\varphi, M_\psi, M_z$  on  $l^2(E)$  respectively. So without loss of generality we can assume that  $\mathcal{K}_2 = l^2(E)$  and  $Q_{21} = M_\varphi, Q_{22} = M_\psi$  and  $V_1 = M_z$  on  $l^2(E)$ . The block matrices of  $M_\varphi, M_\psi, M_z$  are given by

$$M_\varphi = \begin{bmatrix} A & 0 & 0 & \dots \\ B & A & 0 & \dots \\ 0 & B & A & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}, \quad M_\psi = \begin{bmatrix} B^* & 0 & 0 & \dots \\ A^* & B^* & 0 & \dots \\ 0 & A^* & B^* & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$$

$$\text{and } M_z = \begin{bmatrix} 0 & 0 & 0 & \dots \\ I & 0 & 0 & \dots \\ 0 & I & 0 & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}.$$

From now onward we shall consider  $\mathcal{H}$  as a subspace of  $\mathcal{K}$  and  $T_1, T_2, T_3$  on  $\mathcal{H}$  as the restrictions of  $Q_1^*, Q_2^*, V^*$  respectively to  $\mathcal{H}$ .

*Claim 1.*  $\mathcal{D}_{T_3} \subseteq E \oplus \{0\} \oplus \{0\} \oplus \dots \subseteq l^2(E)$ .

*Proof of claim.* Let  $h = h_1 \oplus h_2 \in \mathcal{D}_{T_3} \subseteq \mathcal{H}$ , where  $h_1 \in \mathcal{K}_1$  and  $h_2 = (c_0, c_1, c_2, \dots)^T \in l^2(E)$ . Here  $(c_0, c_1, c_2, \dots)^T$  denotes the transpose of the vector  $(c_0, c_1, c_2, \dots)$ . Since  $T_3(\mathcal{D}_{T_3}) = \{0\}$ , we have that

$$T_3 h = V^* h = V^*(h_1 \oplus h_2) = U_1^* h_1 \oplus M_z^* h_2 = U_1^* h_1 \oplus (c_1, c_2, \dots)^T = 0$$

which implies that  $h_1 = 0$  and  $c_1 = c_2 = \dots = 0$ . This completes the proof of *Claim 1*.

*Claim 2.*  $\text{Ker}(D_{T_3}) \subseteq \{0\} \oplus E \oplus \{0\} \oplus \{0\} \oplus \dots \subseteq l^2(E)$ .

*Proof of claim.* For  $h = h_1 \oplus h_2 \in \text{Ker}(D_{T_3}) \subseteq \mathcal{H}$ , where  $h_1 \in \mathcal{K}_1$  and  $h_2 = (c_0, c_1, c_2, \dots)^T \in l^2(E)$ , we have that

$$D_{T_3}^2 h = (I - T_3^* T_3) h = P_{\mathcal{H}}(I - V V^*) h = P_{\mathcal{H}}(h_1 \oplus h_2 - h_1 \oplus M_z M_z^* h_2) = 0$$

which implies that  $P_{\mathcal{H}}(h_1 \oplus h_2) = P_{\mathcal{H}}(h_1 \oplus M_z M_z^* h_2)$ . Therefore,

$$h_1 \oplus (c_0, c_1, \dots)^T = P_{\mathcal{H}}(h_1 \oplus (0, c_1, c_2, \dots)^T)$$

which further implies that  $\|h_1 \oplus (0, c_1, c_2, \dots)^T\| \geq \|h_1 \oplus (c_0, c_1, c_2, \dots)^T\|$ . Thus  $c_0 = 0$ . Again  $T_3(\text{Ker}(D_{T_3})) \subseteq \mathcal{D}_{T_3}$ . Therefore, for  $h_1 \oplus (0, c_1, c_2, \dots)^T \in \text{Ker}(D_{T_3})$ , we have that

$$T_3(h_1 \oplus (0, c_1, c_2, \dots)^T) = U_1^* h_1 \oplus M_z^* (0, c_1, c_2, \dots)^T = U_1^* h_1 \oplus (c_1, c_2, \dots)^T \in \mathcal{D}_{T_3}.$$

Then by Claim 1,  $h_1 = 0$  and  $c_2 = c_3 = \dots = 0$ . Hence *Claim 2* is established.

Now since  $\mathcal{H} = \mathcal{D}_{T_3} \oplus \text{Ker}(D_{T_3})$ , we can conclude that  $\mathcal{H} \subseteq E \oplus E \oplus \{0\} \oplus \{0\} \oplus \dots \subseteq l^2(E) = \mathcal{K}_2$ . Therefore,  $(M_\varphi^*, M_\psi^*, M_z^*)$  on  $l^2(E)$  is an  $\mathbb{E}$ -co-isometric extension of  $(T_1, T_2, T_3)$ . We now compute the fundamental operators of  $(M_\varphi^*, M_\psi^*, M_z^*)$ .

$$\begin{aligned} & M_\varphi^* - M_\psi M_z^* \\ &= \begin{bmatrix} A^* & B^* & 0 & \dots \\ 0 & A^* & B^* & \dots \\ 0 & 0 & A^* & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} - \begin{bmatrix} B^* & 0 & 0 & \dots \\ A^* & B^* & 0 & \dots \\ 0 & A^* & B^* & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 & I & 0 & \dots \\ 0 & 0 & I & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ &= \begin{bmatrix} A^* & B^* & 0 & \dots \\ 0 & A^* & B^* & \dots \\ 0 & 0 & A^* & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} - \begin{bmatrix} 0 & B^* & 0 & \dots \\ 0 & A^* & B^* & \dots \\ 0 & 0 & A^* & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ &= \begin{bmatrix} A^* & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \end{aligned}$$

Similarly

$$M_\psi^* - M_\varphi M_z^* = \begin{bmatrix} B & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Also

$$D_{M_z^*}^2 = I - M_z M_z^*$$

$$= \begin{bmatrix} I & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Therefore,  $\mathcal{D}_{M_z^*} = E \oplus \{0\} \oplus \{0\} \cdots$  and  $D_{M_z^*}^2 = D_{M_z^*} = I_d$  on  $E \oplus \{0\} \oplus \{0\} \cdots$ . If we set

$$\hat{A}_1 = \begin{bmatrix} A^* & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} B & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \quad (4.1)$$

then

$$M_\varphi^* - M_\psi M_z^* = D_{M_z^*} \hat{A}_1 D_{M_z^*} \text{ and } M_\psi^* - M_\varphi M_z^* = D_{M_z^*} \hat{A}_2 D_{M_z^*}.$$

Therefore,  $\hat{A}_1, \hat{A}_2$  are the fundamental operators of  $(M_\varphi^*, M_\psi^*, M_z^*)$ . Let us denote  $(M_\varphi^*, M_\psi^*, M_z^*)$  by  $(R_1, R_2, W)$ . Therefore,

$$R_1 - R_2^* W = D_W \hat{A}_1 D_W \quad (4.2)$$

$$R_2 - R_1^* W = D_W \hat{A}_2 D_W. \quad (4.3)$$

*Claim 3.*  $\hat{A}_i D_W|_{\mathcal{D}_{T_3}} \subseteq \mathcal{D}_{T_3}$  and  $\hat{A}_i^* D_W|_{\mathcal{D}_{T_3}} \subseteq \mathcal{D}_{T_3}$  for  $i = 1, 2$ .

*Proof of claim.* Clearly  $D_W = D_{M_z^*} = I_d$  on  $\mathcal{D}_W$ . Let  $h_0 = (c_0, 0, 0, \cdots)^T \in \mathcal{D}_{T_3}$ . Then  $\hat{A}_1 D_W h_0 = (A^* c_0, 0, 0, \cdots)^T = M_\varphi^* h_0 = R_1 h_0$ . Since  $R_1|_{\mathcal{H}} = S_1$ ,  $R_1 h_0 \in \mathcal{H}$ . Therefore  $(A^* c_0, 0, 0, \cdots)^T \in \mathcal{D}_{T_3}$  and  $\hat{A}_1 D_W|_{\mathcal{D}_{T_3}} \subseteq \mathcal{D}_{T_3}$ . Similarly we can prove that  $\hat{A}_2 D_W|_{\mathcal{D}_{T_3}} \subseteq \mathcal{D}_{T_3}$ .

We compute the adjoint of  $T_3$ . Let  $(c_0, c_1, 0, \cdots)^T$  and  $(d_0, d_1, 0, \cdots)^T$  be two arbitrary elements in  $\mathcal{H}$  where  $(c_0, 0, 0, \cdots)^T, (d_0, 0, 0, \cdots)^T \in \mathcal{D}_{T_3}$  and  $(0, c_1, 0, \cdots)^T, (0, d_1, 0, \cdots)^T \in \text{Ker}(D_{T_3})$ . Now

$$\begin{aligned} \langle T_3^*(c_0, c_1, 0, \cdots)^T, (d_0, d_1, 0, \cdots)^T \rangle &= \langle (c_0, c_1, 0, \cdots)^T, T_3(d_0, d_1, 0, \cdots)^T \rangle \\ &= \langle (c_0, c_1, 0, \cdots)^T, W(d_0, d_1, 0, \cdots)^T \rangle \\ &= \langle (c_0, c_1, 0, \cdots)^T, (d_1, 0, 0, \cdots)^T \rangle \\ &= \langle c_0, d_1 \rangle_E \\ &= \langle (0, c_0, 0, \cdots)^T, (d_0, d_1, 0, \cdots)^T \rangle. \end{aligned}$$

Therefore

$$T_3^*(c_0, c_1, 0, \dots)^T = (0, c_0, 0, \dots)^T.$$

Now  $h_0 = (c_0, 0, 0, \dots)^T \in \mathcal{D}_{T_3}$  implies that  $T_3^*h_0 = (0, c_0, 0, \dots)^T \in \mathcal{H}$  and  $M_\psi^*(0, c_0, 0, \dots)^T = R_2(0, c_0, 0, \dots)^T = (Ac_0, 0, 0, \dots)^T \in \mathcal{H}$ . In particular,  $(Ac_0, 0, 0, \dots)^T \in \mathcal{D}_{T_3}$ . Therefore  $\hat{A}_1^* D_W h_0 = (Ac_0, 0, 0, \dots)^T \in \mathcal{D}_{T_3}$  and  $\hat{A}_2^* D_W|_{\mathcal{D}_{T_3}} \subseteq \mathcal{D}_{T_3}$ . Similarly we can prove that  $\hat{A}_2^* D_W|_{\mathcal{D}_{T_3}} \subseteq \mathcal{D}_{T_3}$ . Hence *Claim 3* is proved.

*Claim 4.*  $\hat{A}_i|_{\mathcal{D}_{T_3}} = A_i$  and  $\hat{A}_i^*|_{\mathcal{D}_{T_3}} = A_i^*$  for  $i = 1, 2$ .

*Proof of Claim.* It is obvious that  $\mathcal{D}_{T_3} \subseteq \mathcal{D}_W = E \oplus \{0\} \oplus \{0\} \oplus \dots$ . Now since  $W|_{\mathcal{H}} = T_3$  and  $D_W$  is projection onto  $\mathcal{D}_W$ , we have that  $D_W|_{\mathcal{H}} = D_W^2|_{\mathcal{H}} = D_W^2|_{\mathcal{D}_{T_3}} = D_{T_3}^2$ . Therefore,  $D_{T_3}^2$  is a projection onto  $\mathcal{D}_{T_3}$  and  $D_{T_3}^2 = D_{T_3}$ . From (4.2) we have that

$$P_{\mathcal{H}}(R_1 - R_2^*W)|_{\mathcal{H}} = P_{\mathcal{H}}(D_W \hat{A}_1 D_W)|_{\mathcal{H}}. \quad (4.4)$$

Since  $(R_1, R_2, W)$  is an  $\mathbb{E}$ -co-isometric extension of  $(T_1, T_2, T_3)$ , the LHS of (4.4) is equal to  $T_1 - T_2^*T_3$ . Again since  $A_1, A_2$  are the fundamental operators of  $(T_1, T_2, T_3)$ , we have that

$$T_1 - T_2^*T_3 = D_{T_3} A_1 D_{T_3}, \quad A_1 \in \mathcal{L}(\mathcal{D}_{T_3}). \quad (4.5)$$

It is clear that  $T_1 - T_2^*T_3$  is 0 on the ortho-complement of  $\mathcal{D}_{T_3}$ , that is on  $\text{Ker}(D_{T_3})$ . Therefore,

$$T_1 - T_2^*T_3 = P_{\mathcal{D}_{T_3}}(R_1 - R_2^*W)|_{\mathcal{D}_{T_3}} = P_{\mathcal{D}_{T_3}}(D_W \hat{A}_1 D_W)|_{\mathcal{D}_{T_3}}. \quad (4.6)$$

Again since  $D_W|_{\mathcal{D}_{T_3}} = D_{T_3} = I_d$  on  $\mathcal{D}_{T_3}$ , the RHS of (4.6) is equal to  $(D_W \hat{A}_1 D_W)|_{\mathcal{D}_{T_3}}$  and hence

$$T_1 - T_2^*T_3 = (R_1 - R_2^*W)|_{\mathcal{D}_{T_3}} = (D_W \hat{A}_1 D_W)|_{\mathcal{D}_{T_3}} = D_{T_3} \hat{A}_1 D_{T_3}. \quad (4.7)$$

The last identity follows from the fact (*Claim 3*) that  $\hat{A}_1 D_W|_{\mathcal{D}_{T_3}} \subseteq \mathcal{D}_{T_3}$ . By the uniqueness of  $A_1$  we get that  $\hat{A}_1|_{\mathcal{D}_{T_3}} = A_1$ . Also since  $\mathcal{D}_{T_3}$  is invariant under  $\hat{A}_1^*$  by *Claim 3*, we have that  $\hat{A}_1^*|_{\mathcal{D}_{T_3}} = A_1^*$ . Similarly we can prove that  $\hat{A}_2|_{\mathcal{D}_{T_3}} = A_2$  and  $\hat{A}_2^*|_{\mathcal{D}_{T_3}} = A_2^*$ . Thus the proof to *Claim 4* is complete.

Now since  $(M_\varphi, M_\psi, M_z)$  on  $l^2(E)$  is an  $\mathbb{E}$ -isometry,  $M_\varphi$  and  $M_\psi$  commute, that is

$$\begin{aligned} & \begin{bmatrix} A & 0 & 0 & \dots \\ B & A & 0 & \dots \\ 0 & B & A & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} B^* & 0 & 0 & \dots \\ A^* & B^* & 0 & \dots \\ 0 & A^* & B^* & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \\ &= \begin{bmatrix} B^* & 0 & 0 & \dots \\ A^* & B^* & 0 & \dots \\ 0 & A^* & B^* & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \begin{bmatrix} A & 0 & 0 & \dots \\ B & A & 0 & \dots \\ 0 & B & A & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \end{aligned}$$

which implies that

$$\begin{aligned} & \begin{bmatrix} AB^* & 0 & 0 & \dots \\ BB^* + AA^* & AB^* & 0 & \dots \\ BA^* & BB^* + AA^* & AB^* & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix} \\ &= \begin{bmatrix} B^*A & 0 & 0 & \dots \\ A^*A + B^*B & B^*A & 0 & \dots \\ A^*B & A^*A + B^*B & B^*A & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}. \end{aligned}$$

Comparing both sides we obtain the following,

- (1)  $A^*B = BA^*$
- (2)  $A^*A - AA^* = BB^* - B^*B$ .

Therefore from (4.1) we have that

- (1)  $\hat{A}_1\hat{A}_2 = \hat{A}_2\hat{A}_1$
- (2)  $\hat{A}_1^*\hat{A}_1 - \hat{A}_1\hat{A}_1^* = \hat{A}_2^*\hat{A}_2 - \hat{A}_2\hat{A}_2^*$ .

Taking restriction of the above two operator identities to the subspace  $\mathcal{D}_{T_3}$  we get

- (1)  $A_1A_2 = A_2A_1$
- (2)  $A_1^*A_1 - A_1A_1^* = A_2^*A_2 - A_2A_2^*$ .

The proof is now complete. ■

## 5. A COUNTER EXAMPLE

Let  $\mathcal{H}_1 = l^2(E) \oplus l^2(E)$ ,  $E = \mathbb{C}^2$  and let  $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$ . Let  $T_1, T_2, T_3$  on  $\mathcal{H}_1 \oplus \mathcal{H}_1$  be the block operator matrices

$$T_1 = \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix}, T_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } T_3 = \begin{bmatrix} 0 & 0 \\ Y & 0 \end{bmatrix}$$

where

$$J = \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 0 & V \\ I & 0 \end{bmatrix} \text{ on } \mathcal{H}_1 = l^2(E) \oplus l^2(E).$$

Here  $V = M_z$  and  $I = I_d$  on  $l^2(E)$  and  $F$  on  $l^2(E)$  is defined as

$$\begin{aligned} F : l^2(E) &\rightarrow l^2(E) \\ (c_0, c_1, c_2, \dots)^T &\mapsto (F_1 c_0, 0, 0, \dots)^T, \end{aligned}$$

where we choose

$$F_1 = \begin{pmatrix} 0 & \frac{1}{4} \\ 0 & 0 \end{pmatrix}$$

so that  $F_1$  is a non-normal contraction such that  $F_1^2 = 0$ . Clearly  $F^2 = 0$  and  $F^*F \neq FF^*$ . Since  $FV = 0$ ,  $JY = 0$  and thus the product of any two of  $T_1, T_2, T_3$  is equal to 0. Now we unfold the operators  $T_1, T_2, T_3$  and write their block matrices with respect to the decomposition  $\mathcal{H} = l^2(E) \oplus l^2(E) \oplus l^2(E) \oplus l^2(E)$ :

$$T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } T_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix}.$$

We shall prove later that  $(T_1, T_2, T_3)$  is an  $\mathbb{E}$ -contraction and let us assume it for now. Here

$$\begin{aligned} D_{T_3}^2 = I - T_3^* T_3 &= \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & I \\ 0 & 0 & V^* & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} = D_{T_3}. \end{aligned}$$

Clearly  $\mathcal{D}_{T_3} = \{0\} \oplus \{0\} \oplus l^2(E) \oplus l^2(E) = \{0\} \oplus \mathcal{H}_1$  and  $\text{Ker}(D_{T_3}) = l^2(E) \oplus l^2(E) \oplus \{0\} \oplus \{0\} = \mathcal{H}_1 \oplus \{0\}$ . Also for a vector  $k_0 = (h_0, h_1, 0, 0)^T \in \text{Ker}(D_{T_3})$  and for a vector  $k_1 = (0, 0, h_2, h_3)^T \in \mathcal{D}_{T_3}$ ,

$$T_3 k_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix} (h_0, h_1, 0, 0)^T = (0, 0, Vh_1, h_0)^T \in \mathcal{D}_{T_3}$$



and

$$T_3 k_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & V & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix} (0, 0, h_2, h_3)^T = (0, 0, 0, 0)^T.$$

Thus  $(T_1, T_2, T_3)$  satisfies all the conditions of Proposition 4.5. We now compute the fundamental operators  $A_1, A_2$  of  $(T_1, T_2, T_3)$ .

$$T_1 - T_2^* T_3 = T_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & F & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = D_{T_3} A_1 D_{T_3} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} A_1 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

Since  $\mathcal{D}_{T_3} = \{0\} \oplus \mathcal{H}_1$  and  $A_1 \in \mathcal{L}(\mathcal{D}_{T_3})$  we can set

$$A_1 = 0 \oplus \begin{bmatrix} F & 0 \\ 0 & 0 \end{bmatrix} \quad \text{on } \{0\} \oplus \mathcal{H}_1 (= \mathcal{D}_{T_3})$$

so that

$$T_1 - T_2^* T_3 = D_{T_3} A_1 D_{T_3}.$$

Again  $T_1^* T_3 = 0$  as  $X^* V = 0$  and therefore  $T_2 - T_1^* T_3 = 0$ . This shows that the fundamental operator  $A_2$ , for which  $T_2 - T_1^* T_3 = D_{T_3} A_2 D_{T_3}$  holds, has to be equal to 0. Clearly

$$A_1^* A_1 - A_1 A_1^* = 0 \oplus \begin{bmatrix} F^* F - F F^* & 0 \\ 0 & 0 \end{bmatrix} \neq 0 \text{ as } F^* F \neq F F^*$$

but  $A_2^* A_2 - A_2 A_2^* = 0$ . This violets the conclusion of Proposition 4.5 and it is guaranteed that the  $\mathbb{E}$ -contraction  $(T_1^*, T_2^*, T_3^*)$  does not have an  $\mathbb{E}$ -isometric dilation. Since every  $\mathbb{E}$ -unitary dilation is necessarily an  $\mathbb{E}$ -isometric dilation,  $(T_1^*, T_2^*, T_3^*)$  does not have an  $\mathbb{E}$ -unitary dilation.

Now we prove that  $(T_1, T_2, T_3)$  is an  $\mathbb{E}$ -contraction. By Lemma 2.1, it suffices to show that  $\|p(T_1, T_2, T_3)\| \leq \|p\|_{\infty, \mathbb{E}}$ , for any polynomial  $p(x_1, x_2, x_3)$  in the co-ordinates of  $\mathbb{E}$ . Let

$$p(x_1, x_2, x_3) = a_0 + \sum_{i=1}^3 a_i x_i + q(x_1, x_2, x_3),$$

where  $q$  is a polynomial containing only terms of second or higher degree. Now

$$p(T_1, T_2, T_3) = a_0 I + a_1 T_1 + a_3 T_3 = \begin{bmatrix} a_0 I & 0 \\ a_3 Y & a_0 I + a_1 J \end{bmatrix}$$

Since  $Y$  is a contraction and  $\|J\| = \frac{1}{4}$ , it is obvious that

$$\left\| \begin{bmatrix} a_0 I & 0 \\ a_3 Y & a_0 I + a_1 J \end{bmatrix} \right\| \leq \left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + \frac{|a_1|}{4} \end{pmatrix} \right\|.$$

We divide the rest of the proof into two cases.

**Case 1.** When  $|a_0| \leq |a_1|$ .

We show that

$$\left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + \frac{|a_1|}{4} \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} |a_0| & 0 \\ |a_1| + |a_3| & |a_0| \end{pmatrix} \right\|.$$

Let  $\begin{pmatrix} \epsilon \\ \delta \end{pmatrix}$  be a unit vector in  $\mathbb{C}^2$  such that

$$\left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + \frac{|a_1|}{4} \end{pmatrix} \right\| = \left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + \frac{|a_1|}{4} \end{pmatrix} \begin{pmatrix} \epsilon \\ \delta \end{pmatrix} \right\|.$$

Without loss of generality we can choose  $\epsilon, \delta \geq 0$  because

$$\left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + \frac{|a_1|}{4} \end{pmatrix} \begin{pmatrix} \epsilon \\ \delta \end{pmatrix} \right\|^2 = |a_0 \epsilon|^2 + \left| |a_3 \epsilon| + \left( |a_0| + \frac{|a_1|}{4} \right) \delta \right|^2$$

and if we replace  $\begin{pmatrix} \epsilon \\ \delta \end{pmatrix}$  by  $\begin{pmatrix} |\epsilon| \\ |\delta| \end{pmatrix}$  we see that

$$\left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + \frac{|a_1|}{4} \end{pmatrix} \begin{pmatrix} |\epsilon| \\ |\delta| \end{pmatrix} \right\|^2 \geq \left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + \frac{|a_1|}{4} \end{pmatrix} \begin{pmatrix} \epsilon \\ \delta \end{pmatrix} \right\|^2.$$

So, assuming  $\epsilon, \delta \geq 0$  we get

$$\begin{aligned}
& \left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + \frac{|a_1|}{4} \end{pmatrix} \begin{pmatrix} \epsilon \\ \delta \end{pmatrix} \right\|^2 \\
&= |a_0\epsilon|^2 + \left\{ |a_3\epsilon| + \left( |a_0| + \frac{|a_1|}{4} \right) \delta \right\}^2 \\
&= |a_0\epsilon|^2 + |a_3\epsilon|^2 + \left\{ |a_0|^2 + \frac{|a_0a_1|}{2} + \frac{|a_1|^2}{16} \right\} \delta^2 + 2|a_3| \left( |a_0| + \frac{|a_1|}{4} \right) \epsilon\delta \\
&= \{(|a_0|^2 + |a_3|^2)\epsilon^2 + |a_0|^2\delta^2 + 2|a_0a_3|\epsilon\delta\} + \left\{ \frac{|a_1|^2}{16} + \frac{|a_0a_1|}{2} \right\} \delta^2 + \frac{|a_1a_3|}{2}\epsilon\delta.
\end{aligned} \tag{5.1}$$

Again

$$\begin{aligned}
& \left\| \begin{pmatrix} |a_0| & 0 \\ |a_1| + |a_3| & |a_0| \end{pmatrix} \begin{pmatrix} \epsilon \\ \delta \end{pmatrix} \right\|^2 \\
&= |a_0\epsilon|^2 + \{(|a_1| + |a_3|)\epsilon + |a_0|\delta\}^2 \\
&= |a_0|^2\epsilon^2 + \{|a_1|^2 + |a_3|^2 + 2|a_1a_3|\}\epsilon^2 + 2|a_0|(|a_1| + |a_3|)\epsilon\delta + |a_0|^2\delta^2 \\
&= \{(|a_0|^2 + |a_3|^2)\epsilon^2 + |a_0|^2\delta^2 + 2|a_0a_3|\epsilon\delta\} + (|a_1|^2\epsilon^2 + 2|a_0a_1|\epsilon\delta) + 2|a_1a_3|\epsilon^2.
\end{aligned} \tag{5.2}$$

We now compare (5.1) and (5.2). If  $\epsilon \geq \delta$  then

$$(|a_1|^2\epsilon^2 + 2|a_0a_1|\epsilon\delta) + 2|a_1a_3|\epsilon^2 \geq \left( \frac{|a_1|^2}{16} + \frac{|a_0a_1|}{2} \right) \delta^2 + \frac{|a_1a_3|}{2}\epsilon\delta$$

Therefore, it is evident from (5.1) and (5.2) that

$$\left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + \frac{|a_1|}{4} \end{pmatrix} \begin{pmatrix} \epsilon \\ \delta \end{pmatrix} \right\|^2 \leq \left\| \begin{pmatrix} |a_0| & 0 \\ |a_1| + |a_3| & |a_0| \end{pmatrix} \begin{pmatrix} \epsilon \\ \delta \end{pmatrix} \right\|^2.$$

If  $\epsilon < \delta$  we consider the unit vector  $\begin{pmatrix} \delta \\ \epsilon \end{pmatrix}$  and it suffices if we show that

$$\left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + \frac{|a_1|}{4} \end{pmatrix} \begin{pmatrix} \epsilon \\ \delta \end{pmatrix} \right\|^2 \leq \left\| \begin{pmatrix} |a_0| & 0 \\ |a_1| + |a_3| & |a_0| \end{pmatrix} \begin{pmatrix} \delta \\ \epsilon \end{pmatrix} \right\|^2.$$

A computation similar to (5.2) gives

$$\begin{aligned}
& \left\| \begin{pmatrix} |a_0| & 0 \\ |a_1| + |a_3| & |a_0| \end{pmatrix} \begin{pmatrix} \delta \\ \epsilon \end{pmatrix} \right\|^2 \\
&= |a_0|^2 \delta^2 + \{|a_1|^2 + |a_3|^2 + 2|a_1 a_3|\} \delta^2 + 2|a_0|(|a_1| + |a_3|) \epsilon \delta + |a_0|^2 \epsilon^2 \\
&= \{|a_0|^2(\epsilon^2 + \delta^2) + 2|a_0 a_3| \epsilon \delta\} + \{|a_1|^2 + |a_3|^2 + 2|a_1 a_3|\} \delta^2 + 2|a_0 a_1| \epsilon \delta \\
&= \{|a_0|^2 + 2|a_0 a_3| \epsilon \delta\} + \{|a_1|^2 + |a_3|^2 + 2|a_1 a_3|\} \delta^2 + 2|a_0 a_1| \epsilon \delta. \quad (5.3)
\end{aligned}$$

In the last equality we used the fact that  $|\epsilon|^2 + |\delta|^2 = 1$ . Again from (5.1) we have

$$\begin{aligned}
& \left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + \frac{|a_1|}{4} \end{pmatrix} \begin{pmatrix} \epsilon \\ \delta \end{pmatrix} \right\|^2 \\
&= \{|a_0|^2(\epsilon^2 + \delta^2) + 2|a_0 a_3| \epsilon \delta\} + \left\{ |a_3|^2 \epsilon^2 + \frac{|a_1 a_3|}{2} \epsilon \delta \right\} + \left\{ \frac{|a_1|^2}{16} + \frac{|a_0 a_1|}{2} \right\} \delta^2 \\
&\leq \{|a_0|^2(\epsilon^2 + \delta^2) + 2|a_0 a_3| \epsilon \delta\} + \left\{ |a_3|^2 \epsilon^2 + \frac{|a_1 a_3|}{2} \epsilon \delta \right\} + \left\{ \frac{|a_1|^2}{16} + \frac{|a_1|^2}{2} \right\} \delta^2 \\
&= \{|a_0|^2 + 2|a_0 a_3| \epsilon \delta\} + \left\{ \frac{9|a_1|^2}{16} \delta^2 + |a_3|^2 \epsilon^2 + \frac{|a_1 a_3|}{2} \epsilon \delta \right\} \quad (5.4)
\end{aligned}$$

The last inequality follows from the fact that  $|a_0| \leq |a_1|$ . Since  $\epsilon < \delta$  we can conclude from (5.3) and (5.4) that

$$\left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + \frac{|a_1|}{4} \end{pmatrix} \begin{pmatrix} \epsilon \\ \delta \end{pmatrix} \right\|^2 \leq \left\| \begin{pmatrix} |a_0| & 0 \\ |a_1| + |a_3| & |a_0| \end{pmatrix} \begin{pmatrix} \delta \\ \epsilon \end{pmatrix} \right\|^2.$$

Therefore,

$$\|p(T_1, T_2, T_3)\| \leq \left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + \frac{|a_1|}{4} \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} |a_0| & 0 \\ |a_1| + |a_3| & |a_0| \end{pmatrix} \right\|.$$

A classical result of Caratheodory and Fejér states that

$$\inf \|b_0 + b_1 z + r(z)\|_{\infty, \mathbb{D}} = \left\| \begin{pmatrix} b_0 & 0 \\ b_1 & b_0 \end{pmatrix} \right\|,$$

where the infimum is taken over all polynomials  $r(z)$  in one variable which contain only terms of degree two or higher. For an elegant proof to this result, see Sarason's seminal paper [19], where the result is derived as a consequence of the classical commutant lifting theorem of

Sz.-Nagy and Foias (see [21]). Using this fact we have that

$$\begin{aligned}
\|p(T_1, T_2, T_3)\| &\leq \left\| \begin{pmatrix} |a_0| & 0 \\ |a_1| + |a_3| & |a_0| \end{pmatrix} \right\| \\
&= \inf \| |a_0| + (|a_1| + |a_3|)z + r(z) \|_{\infty, \mathbb{D}} \\
&\leq \inf \| |a_0| + |a_1|x_1 + |a_3|x_3 + r_1(x_1, x_2, x_3) \|_{\infty, \Lambda} \quad (5.5) \\
&\leq \inf \| |a_0| + |a_2| + |a_1|x_1 + |a_3|x_3 + r_1(x_1, x_2, x_3) \|_{\infty, \Lambda} \quad (5.6) \\
&= \inf \| |a_0| + |a_1|x_1 + |a_2|x_2 + |a_3|x_3 + r_1(x_1, x_2, x_3) \|_{\infty, \Lambda} \\
&\leq \| a_0 + a_1x_1 + a_2x_2 + a_3x_3 + q(x_1, x_2, x_3) \|_{\infty, \Lambda} \quad (5.7) \\
&\leq \| a_0 + a_1x_1 + a_2x_2 + a_3x_3 + q(x_1, x_2, x_3) \|_{\infty, \overline{\mathbb{E}}} \\
&= \| p(x_1, x_2, x_3) \|_{\infty, \overline{\mathbb{E}}}.
\end{aligned}$$

Here  $\Lambda = \{(x, 1, x) : x \in \mathbb{D}\} \subseteq \overline{\mathbb{E}}$  (by choosing  $\beta_1 = 0, \beta_2 = 1$  in Theorem 1.1) and  $r(z)$  and  $r_1(x_1, x_2, x_3)$  range over polynomials of degree two or higher. The inequality (5.5) was obtained by putting  $x_1 = x_3 = z$  and  $x_2 = 1$  which makes the set of polynomials  $|a_0| + |a_1|x_1 + |a_3|x_3 + r_1(z_1, z_2, z_3)$ , a subset of the set of polynomials  $|a_0| + (|a_1| + |a_3|)z + r(z)$ . The infimum taken over a subset is always bigger than or equal to the infimum taken over the set itself. We obtained the inequality (5.6) by applying a similar argument because we can extract the polynomial  $|a_2|x_2^2$  from the set  $r_1(x_1, x_2, x_3)$  and  $|a_2|x_2^2 = |a_2|$  when  $x_2 = 1$ . The inequality (5.7) was obtained by choosing  $r_1(x_1, x_2, x_3)$  in particular to be equal to

$$(a_0 - |a_0| + a_2 - |a_2|)x_2^2 + (a_1 - |a_1|)x_1x_2 + (a_3 - |a_3|)x_2x_3 + q(x_1, x_2, x_3).$$

**Case 2.** When  $|a_0| > |a_1|$ .

It is obvious from Case 1 that

$$\left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + \frac{|a_1|}{4} \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} |a_0| & 0 \\ |a_3| & |a_0| + \frac{|a_0|}{4} \end{pmatrix} \right\| \leq \left\| \begin{pmatrix} |a_0| & 0 \\ |a_0| + |a_3| & |a_0| \end{pmatrix} \right\|.$$

Therefore,

$$\begin{aligned}
\|p(T_1, T_2, T_3)\| &\leq \left\| \begin{pmatrix} |a_0| & 0 \\ |a_0| + |a_3| & |a_0| \end{pmatrix} \right\| \\
&= \inf \| |a_0| + (|a_0| + |a_3|)z + r(z) \|_{\infty, \mathbb{D}} \\
&\leq \inf \| |a_0| + |a_0|x_1 + |a_3|x_3 + r_1(x_1, x_2, x_3) \|_{\infty, \Lambda} \\
&\leq \inf \| |a_0| + |a_2| + |a_0|x_1 + |a_3|x_3 + r_1(x_1, x_2, x_3) \|_{\infty, \Lambda} \\
&= \inf \| |a_0| + |a_0|x_1 + |a_2|x_2 + |a_3|x_3 + r_1(x_1, x_2, x_3) \|_{\infty, \Lambda} \\
&\leq \|a_0 + a_1x_1 + a_2x_2 + a_3x_3 + q(x_1, x_2, x_3)\|_{\infty, \Lambda} \quad (5.8) \\
&\leq \|a_0 + a_1x_1 + a_2x_2 + a_3x_3 + q(x_1, x_2, x_3)\|_{\infty, \mathbb{E}} \\
&= \|p(x_1, x_2, x_3)\|_{\infty, \mathbb{E}}.
\end{aligned}$$

Here all notations used are as same as they were in Case 1 and we obtained the inequality (5.8) by choosing  $r_1(x_1, x_2, x_3)$  in particular to be equal to

$$(a_0 - |a_0| + a_2 - |a_2|)x_2^2 + (a_1 - |a_0|)x_1x_2 + (a_3 - |a_3|)x_2x_3 + q(x_1, x_2, x_3).$$

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## REFERENCES

- [1] A. A. Abouhajar, M. C. White and N. J. Young, A Schwarz lemma for a domain related to  $\mu$ -synthesis, *J. Geom. Anal.* 17 (2007), 717 – 750.
- [2] A. A. Abouhajar, M. C. White and N. J. Young, Corrections to 'A Schwarz lemma for a domain related to  $\mu$ -synthesis', available online at <http://www1.maths.leeds.ac.uk/~nicholas/abstracts/correction.pdf>
- [3] J. Agler, Rational dilation on an annulus, *Ann. of Math.* 121 (1985), 537 – 563.
- [4] J. Agler, J. Harland and B. J. Raphael, Classical function theory, operator dilation theory, and machine computation on multiply-connected domains, *Mem. Amer. Math. Soc.* 191 (2008), 289 – 312.
- [5] J. Agler and N. J. Young, A commutant lifting theorem for a domain in  $\mathbb{C}^2$  and spectral interpolation, *J. Funct. Anal.* 161 (1999), 452 – 477.
- [6] H. Alexander and J. Wermer, Several complex variables and Banach algebras, *Graduate Texts in Mathematics*, 35; 3rd Edition, Springer, (1997).
- [7] T. Ando, On a pair of commutative contractions, *Acta Sci Math* 24 (1963), 88 – 90.
- [8] W. Arveson, Subalgebras of  $C^*$ -algebras II, *Acta Math.*, 128 (1972), 271 – 308.

- [9] T. Bhattacharyya, The tetrablock as a spectral set, *Indiana Univ. Math. Jour.*, 63 (2014), 1601 – 1629.
- [10] T. Bhattacharyya, S. Pal and S. Shyam Roy, Dilations of  $\Gamma$ -contractions by solving operator equations, *Adv. Math.* 230 (2012), 577 – 606.
- [11] T. Bhattacharyya and H. Sau, Normal boundary dilations in two inhomogeneous domains, *arXiv:1311.1577v1 [math.FA]*.
- [12] M. A. Dritschel and S. McCullough, The failure of rational dilation on a triply connected domain, *J. Amer. Math. Soc.* 18 (2005), 873 – 918.
- [13] A. Edigarian and W. Zwonek, Schwarz lemma for the tetrablock, *Bull. Lond. Math. Soc.* 41 (2009), no. 3, 506 – 514.
- [14] A. Edigarian, L. Kosinski and W. Zwonek, The Lempert theorem and the tetrablock, *J. Geom. Anal.* 23 (2013), 1818 – 1831.
- [15] B. Fuglede, A commutativity theorem for normal operators, *Nat. Acad. Sci.* 36 (1950), 35 – 40.
- [16] S. Pal, From Stinespring dilation to Sz.-Nagy dilation on the symmetrized bidisc and operator models, *New York J. Math.*, 20 (2014), 645–664.
- [17] S. Pal, Subvarieties of the tetrablock and von-Neumann’s inequality, To appear in *Indiana Univ. Math. Jour.*, Available at *arXiv:1405.2436v4 [math.FA]*, 31 Mar, 2015.
- [18] S. Parrott, Unitary dilations for commuting contractions, *Pacific J. Math.*, 34 (1970), 481 – 490.
- [19] D. Sarason, Generalized interpolation in  $H^\infty$ , *Trans. Amer. Math. Soc.*, 127 (1967), 179 – 203.
- [20] B. Sz.-Nagy, Sur les contractions de l’espace de Hilbert, *Acta Sci. Math.*, 15 (1953), 87 – 92.
- [21] B. Sz.-Nagy, C. Foias, H. Bercovici and L. Kerchy, Harmonic analysis of operators on Hilbert space, Universitext, *Springer, New York*, 2010.
- [22] N. J. Young, The automorphism group of the tetrablock, *J. London Math. Soc.*, 77 (2008), 757 – 770.
- [23] W. Zwonek, Geometric properties of the tetrablock, *Arch. Math.* 100 (2013), 159 – 165.

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